

# Gamma Generalized Extended Inverse Exponential Distribution: A Novel Distribution in Modeling COVID-19 Cases in Nigeria

Ogunde AA<sup>1\*</sup>; Chukwu AU<sup>2</sup>; Oseghale OI<sup>3</sup> and Nwanyibuife OB<sup>4</sup>

<sup>1,2</sup> Department of Statistics, University of Ibadan, Ibadan, Oyo State, Nigeria.

<sup>3</sup>Department of Mathematics and Statistics, Joseph Babalola University, Arakeji, Osun State, Nigeria.

<sup>4</sup>Department of Statistics, the Federal University of Technology Owerri, Imo State, Nigeria.

ORCID ID: 0000-0001-8708-8612

## ABSTRACT

In this study, we developed a novel distribution called Gamma Inverse Exponential (GIE) distribution, which has proved to be a more flexible distribution in modeling COVID-19 case fatality in Nigeria. We studied some statistical properties of the new distribution, which include: moments, incomplete moments, quantile function, Renyi entropy, and mean deviation. A real-life data application to a number of reported cases of COVID-19 infection between March 2019 to 2021 shows that the GIE distribution has a better fit than some competing distributions in fitting the data. Time series analysis of the COVID-19 data is also considered.

**Keywords:** COVID-19; Characteristics Function; Mean Residual Function; Case Fatality; Gamma Inverse Exponential Distribution.

## INTRODUCTION

In probability theory, the extreme value (EV) distribution is a family of continuous distributions developed within the framework of extreme value theory, which includes the popular Gumbel, Frechet and Weibull distributions which are respectively known as type I, II and III extreme value distributions. The EV distributions sometimes exist as a limiting distribution for the minimum or maximum of a sample of independent, identically distributed random variables. Extreme value is the theory of modeling and measuring events which occur with a very small probability. These distributions and their generalized forms can be applied in finance, material sciences, telecommunications, economics and many others.

Frechet distribution, because of its heavy tail, has been applied in the modeling of market returns which is related to finance [1]. The cumulative distribution function (CDF) of the standard Frechet distribution is

$$F(x) = e^{\left(\frac{\lambda}{x}\right)^\rho}, \quad x > 0, \lambda, \rho > 0 \quad (1.1)$$

Where  $\lambda$  is a scale parameter, and  $\rho$  is a shape parameter

By letting  $\rho = 1$ , we obtain another distribution known as Inverse Exponential (IE) distribution which CDF is given by

$$F(x) = e^{-\frac{\lambda}{x}}, \quad x > 0, \lambda > 0 \quad (1.2)$$

The associated probability density function (PDF) to (1.2) is given by

$$f(x) = \lambda x^{-2} e^{-\frac{\lambda}{x}}, \quad x > 0, \lambda > 0 \quad (1.3)$$

Here, the  $\lambda$  is a scalar parameter.

In many real-life situations, the classical distribution does not give a sufficient fit to lifetime data, most especially when the data exhibit different shapes of the hazard function. Therefore, various generators are proposed to produce a new model with improved modeling potential [2-13].

**THE GAMMA INVERSE EXPONENTIAL (GIE) DISTRIBUTION**

Zografos and Balakrishnan [12] and Ristic and Balakrishnan [13] proposed a family of univariate distributions generated using gamma random variables. Given any baseline CDF  $F(x)$ , and  $x \in R$ , they defined the gamma-G distribution with an extra shape parameter  $\sigma > 0$  and PDF  $g(x)$  and CDF  $G(x)$  given by

$$g(x) = \frac{1}{\Gamma(\sigma)} [-\log\{1 - F(x)\}]^{\sigma-1} f(x) \quad (2.1)$$

And

$$G(x) = \frac{\gamma(\sigma, -\log[1 - F(x)])}{\Gamma(\sigma)} = \frac{1}{\Gamma(\sigma)} \int_0^{-\log\{1-F(x)\}} t^{\sigma-1} e^{-t} dt, \quad (2.2)$$

respectively, where  $f(x) = dF(x)/dx$ ,  $\Gamma(\sigma) = \int_0^\infty t^{\sigma-1} e^{-t} dt$  and  $\gamma(\sigma, y) = \int_0^y t^{\sigma-1} e^{-t} dt$  are the gamma and the incomplete gamma functions. The shape parameter  $\sigma$  controls skewness and kurtosis through the tail weight

Putting (1.1) and (1.2) in (2.1), we obtain the PDF of the GIE distribution given by

$$g(x) = \frac{\lambda\sigma}{\Gamma(\sigma)} x^{-2} e^{-\frac{\lambda}{x}} \left[ -\log\left\{ 1 - e^{-\frac{\lambda}{x}} \right\} \right]^{\sigma-1} \quad (2.3)$$

And the corresponding CDF to (2.3) is given by

$$G(x) = \frac{\gamma\left(\sigma, -\log\left[1 - e^{-\frac{\lambda}{x}}\right]\right)}{\Gamma(\sigma)} \quad (2.4)$$

An expression for the survival and the hazard function is, respectively as

$$S(x) = 1 - \frac{\gamma\left(\sigma, -\log\left[1 - e^{-\frac{\lambda}{x}}\right]\right)}{\Gamma(\sigma)}, \quad (2.5)$$

And

$$h(x) = \frac{\lambda\sigma x^{-2} e^{-\frac{\lambda}{x}} \left[ -\log\left\{ 1 - e^{-\frac{\lambda}{x}} \right\} \right]}{\Gamma(\sigma) - \gamma\left(\sigma, -\log\left[1 - e^{-\frac{\lambda}{x}}\right]\right)}. \quad (2.6)$$

The graphs of the density and the hazard function are given in Figures 1 and 2 for various values of the parameters.

**QUANTILE FUNCTION OF GIE DISTRIBUTION**

From Nadarajah et al. [9], we can generate GIE random variables from the quantile function given by

$$F^{-1}(u) = \lambda \left[ -\log\left( 1 - e^{(-Q^{-1}(\sigma, 1-u))} \right) \right] \quad (2.7)$$

Where  $Q^{-1}(\sigma, u)$  denote the inverse function of  $Q(\sigma, x) = 1 - \frac{\gamma(\sigma, x)}{\Gamma(\sigma, x)}$ ; setting  $u = 0.5$

in (2.7), we obtain the median ( $M$ ) of GIE distribution as

$$M = \lambda \left[ -\log\left( 1 - e^{(-Q^{-1}(\sigma, 0.5))} \right) \right] \quad (2.8)$$

**PROPERTIES OF GIE DISTRIBUTION**

Here, we obtain an expression for the moment, incomplete moment, mean deviation and the entropy of the GIE model.

**MOMENTS AND INCOMPLETE MOMENTS OF GIE DISTRIBUTION**

We now obtain the  $r^{th}$  moment about zero of  $X$ , say  $E(X^r)$ . The  $r^{th}$  moment of  $X \sim GIE(\lambda, \sigma)$  is found by

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \tag{3.0}$$

Substitute from Equation (2.3) into Equation (3.0), and we will get the  $r^{th}$  moment as follows.

$$E(X^r) = \lambda^r \Gamma(1-r) \sum_{i,l=0}^{\infty} \frac{(-1)^{l+1} (i+1) V_i}{(1+l)^{(1-r)}} \binom{i}{l} \tag{3.1}$$

Where

$$= \frac{(-1)^i}{(i+1)!} \sum_{k=0}^{\infty} \frac{(\sigma+k)}{\Gamma(\sigma+k-i)} \Gamma(\sigma+k)$$

Next, we derive a simple formula for the  $r^{th}$  incomplete moment of  $X$ , say  $m_r(t) = E(X_r | X < t)$ . From equation (2.3), we obtain

$$m_r(t) = \lambda^r \Gamma(1-r) \sum_{i,l=0}^{\infty} \frac{(-1)^{l+1} (i+1) V_i}{(1+l)^{(1-r)}} \binom{i}{l} \left\{ [(1+l)\lambda]^{-(1-r)} - \gamma \left[ 1-r, (1+l) \frac{\lambda}{t} \right] \right\} \tag{3.2}$$

**MEAN DEVIATIONS**

The mean deviations about the mean ( $\delta_1(X) = E(|X - \mu'_1|)$ ) and about the median ( $\delta_2(X) = E(|X - M|)$ ) of  $X$  can be represented as:

$$\frac{1}{1-\rho} \log \left\{ \frac{(\sigma-1)\rho\lambda^\rho}{[\Gamma(\sigma)]^\rho} \sum_{k,l,m} (-1)^{k+l+m+1} \frac{\binom{k-(\sigma-1)\rho}{k} \binom{k}{m} p_{m,k}}{(\sigma-1)\rho - m} \frac{\Gamma[(\sigma-1)\rho + 1] \Gamma(w)}{l! [(k+l+\sigma w)]^w} \right\} \times \Gamma[(\sigma-1)\rho + 1 - l] \tag{3.5}$$

Where,

$$w = \rho[(\sigma-1)\rho + 1 - l] \text{ and } p_{m,k} = k^{-1} \sum_{j=0}^k \frac{(-1)^j [j(m+1) - k]}{(j+1)} p_{m,k-j}$$

For  $k = 1, 2, \dots$ , and  $p_{m,0} = 1$

**ESTIMATION**

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the GIE distribution from complete samples only. Let  $x_1, \dots$

$$\delta_1(X) = 2\mu'_1 F(\mu'_1) - 2m_1(\mu'_1) \text{ and } \delta_2(X) = \mu'_1 - 2m_1(M) \tag{3.3}$$

Respectively, where  $\mu'_1 = E(X)$  can be obtained from (3.1),  $M = \text{median}(X)$  is the median given in (2.8).  $F(\mu'_1)$  can be easily obtained from the CDF (2.2), and  $m_1(t) = \int_0^t x f(x) dx$  is the first incomplete moment obtained from (3.2) by setting  $r = 1$ .

Applications of these equations can be used to obtain an explicit expression for the Bonferroni and Lorenz curves defined for a given probability  $\pi$  by  $B(\pi) = \frac{m_1(h)}{\pi\mu'_1}$  and  $L(\pi) = \frac{m_1(h)}{\mu'_1}$ , respectively, where

$h = F^{-1}(\pi)$  is the GIE function at  $\pi$  defined from (2.7).

**RENYI ENTROPY**

The entropy of a random variable  $X$  with density function  $f(x)$  is a measure of the variation of the uncertainty. For any real parameter  $\rho > 0$  and  $\rho = 1$ , the Renyi entropy is given by

$$I_R(\rho) = \frac{1}{1-\rho} \log \int_0^\infty f^\rho(x) dx \tag{3.4}$$

Putting equation (2.3) in (2.4), we have an expression for the moments of GIE distribution as

,  $x_n$  be a random sample of size  $n$  from the GIE ( $\lambda, \sigma$ ) distribution. The log-likelihood function can be expressed as

$$l(\lambda, \sigma) = -n \log \Gamma(\sigma) + n \log(\lambda\sigma) - \lambda \sum_{i=1}^n x_i^{-1} - 2 \sum_{i=1}^n \log(x_i) + (\sigma-1) \sum_{i=1}^n \left[ \log \left( -\log \left[ 1 - e^{-\frac{\lambda}{x_i}} \right] \right) \right]$$

The element of the score vector for GIE model is given as

$$\frac{\partial l}{\partial \lambda} = \frac{[n - \lambda \sum_{i=1}^n (x_i^{-1})]}{\lambda} + (\sigma - 1) \sum_{i=1}^n \left[ \frac{\frac{\sigma}{x_i} e^{\frac{\sigma}{x_i}}}{\lambda \left(1 - e^{\frac{\sigma}{x_i}}\right) \log \left(1 - e^{\frac{\sigma}{x_i}}\right)} \right] \quad (4.1)$$

$$\frac{\partial l}{\partial \sigma} = n\psi(\sigma) + \sum_{i=1}^n \log \left( -\log \left[ 1 - e^{\frac{\sigma}{x_i}} \right] \right) \quad (4.2)$$

**APPLICATION**

In this section, we present the usefulness of the GIE distribution by applying it to the case fatality COVID-19 data set, Nigeria's experience between the months of March 2020 to December 2021 and compare the performance of the model with its sub-model. Now, we apply the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn criterion (HQIC), and Kolmogorov-Smirnov(K) goodness of fit test to verify which distribution fits better to these data. In

general, the smaller the values of the statistics AIC, CAIC, HQIC and KS, the better the fit to the data. Table I contains the exploratory data analysis for the COVID-19 data. Table 2 shows the measure of goodness of fit test. Table 3 presents the Auto-Regressive Integrated Moving Average (ARIMA) model for the COVID-19 data. The graph of the daily number of reported cases is drawn in Figure 4.

**CONCLUSION**

The Gamma Inverse Exponential model is considered a better model when compared to its sub-model in modeling the COVID-19 data because it returns the minimum values of information criteria. ARIMA (1, 0, 1) is the best model under the ARIMA scheme, and the point forecast obtained indicates that if a drastic step is not taken by January 2023, the monthly reported cases of COVID-19 patients will be 23008.

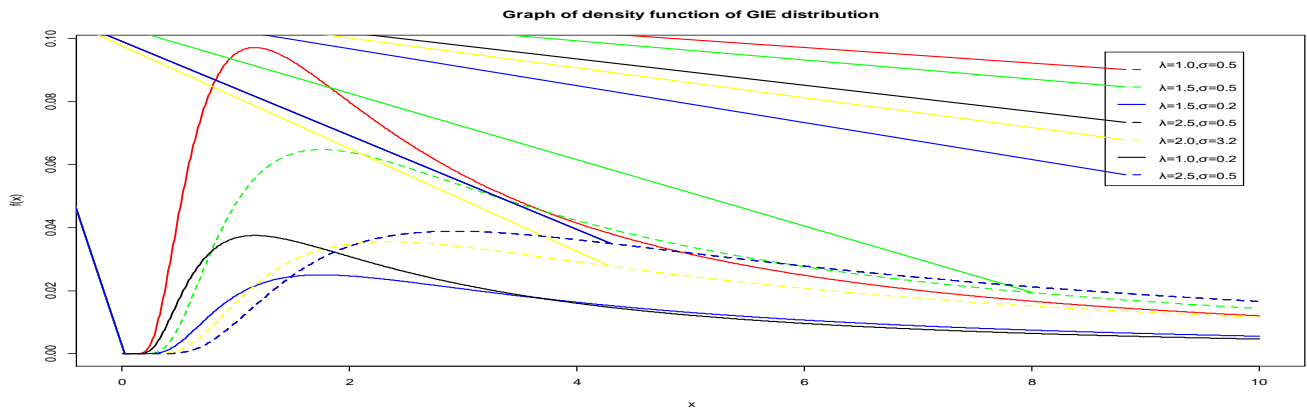


Figure 1: Graph of the density function of GIE distribution

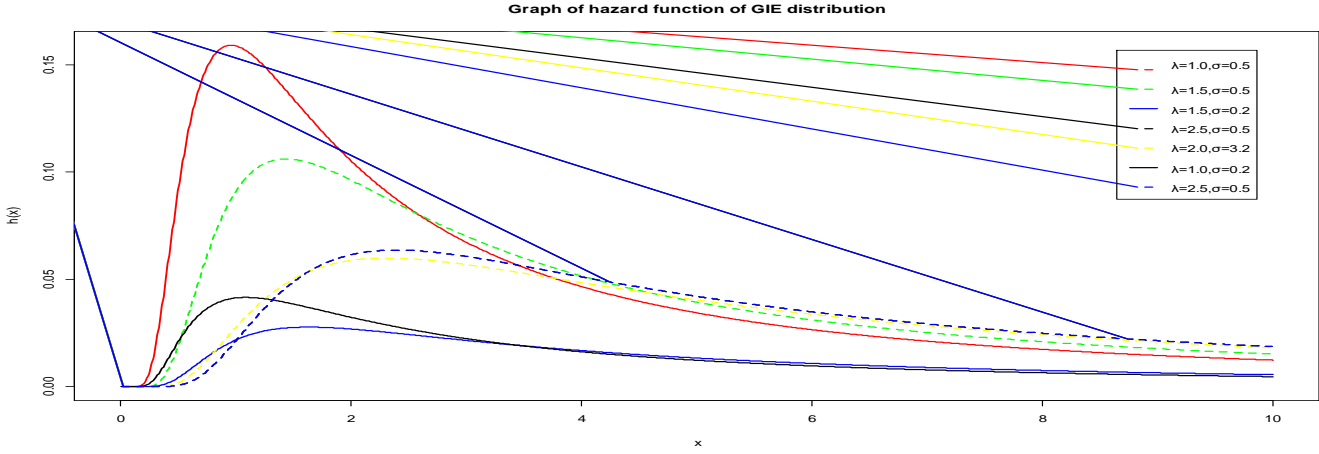


Figure 2: Graph of the hazard function of GIE distribution

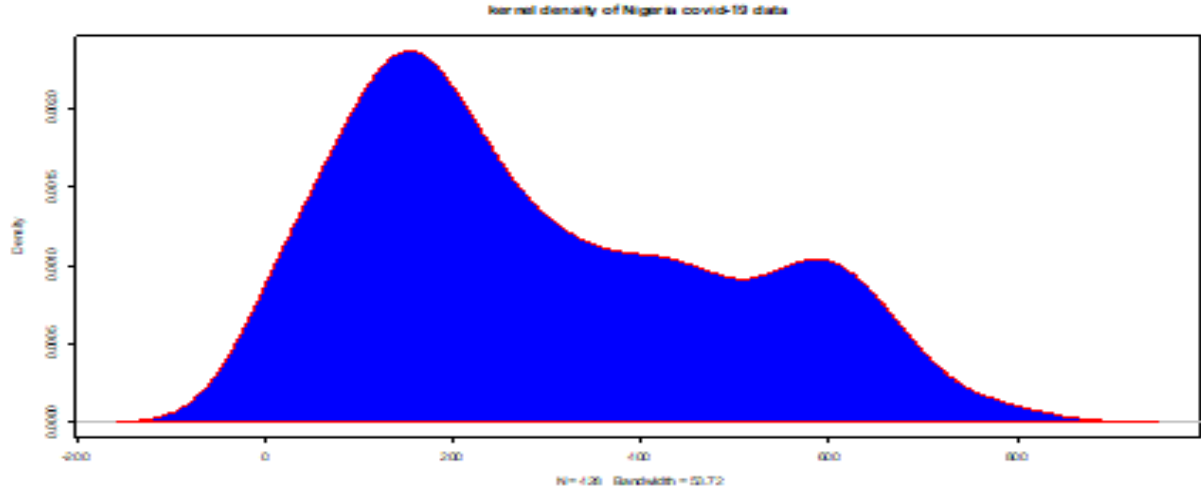


Figure 3: Kernel density curve for COVID-19 data

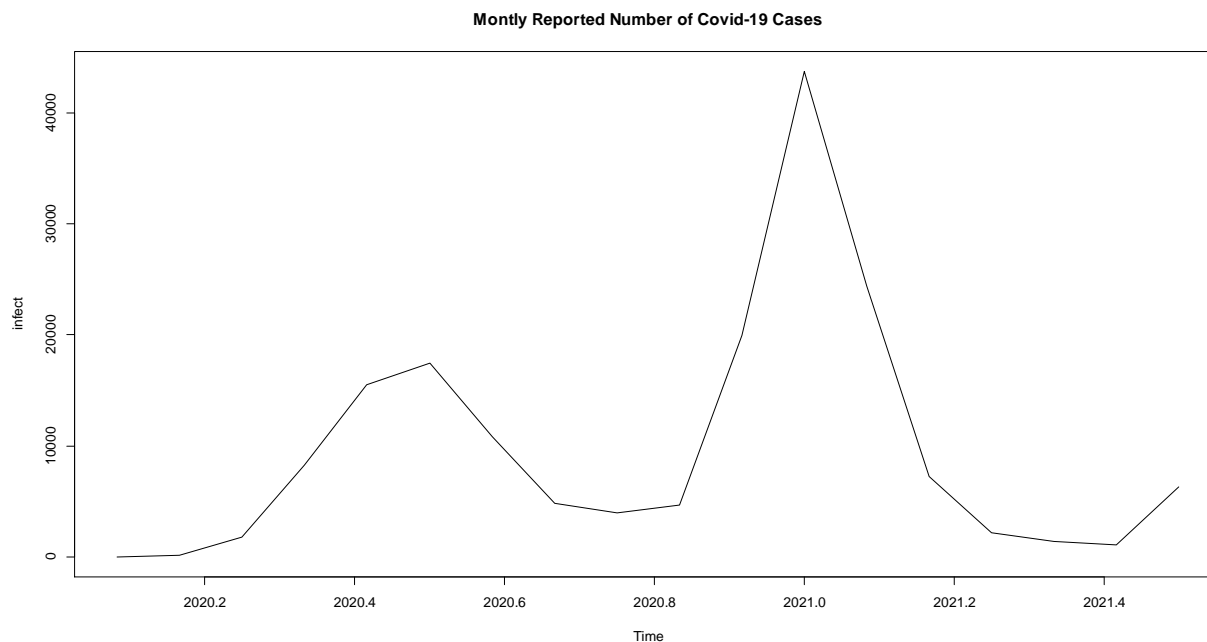


Figure 4: Graph of daily reported cases of COVID-19 in Nigeria.

Table 1: Exploratory data Analysis of COVID-19 data

Min.	$q_1$	$q_2$	Med.	Mean	Max.	Var.	Skew.	Kurt.
1.0	86.0	499.5	203.0	345.4	2314.0	139613	2.0	7.3

Table 2: Measures of goodness of fit of GIE model

Model	$\lambda$	$\sigma$	$-l$	AIC	CAIC	HQIC	KS
GIE	20.05 (0.93)	1.46 (0.04)	4403.42	8810.85	8810.87	8814.29	0.2347
IE	5.72 (2.23)	– (–)	4585.35	9172.72	9172.71	9174.42	0.3090

Table 3: Models with information criterion

<i>ARIMA models</i>	<i>Value of mean</i>	<i>AIC</i>
<i>ARIMA(2,0,2)</i>	<i>With nozero mean</i>	376.3729
<i>ARIMA(0,0,0)</i>	<i>With non – zero mean</i>	506.9031
<i>ARIMA(1,0,0)</i>	<i>With non – zero mean</i>	391.5391
<i>ARIMA(0,0,1)</i>	<i>With non – zero mean</i>	398.7286
<i>ARIMA(0,0,0)</i>	<i>With zero mean</i>	504.9151
<i>ARIMA(1,0,2)</i>	<i>With non – zero mean</i>	374.1229
<i>ARIMA(0,0,2)</i>	<i>with non – zero mean</i>	394.9124
<i>ARIMA(1,0,1)</i>	<i>with non – zero mean</i>	384.8265
<i>ARIMA(1,0,3)</i>	<i>with non – zero mean</i>	374.6944
<i>ARIMA(0,0,2)</i>	<i>with non – zero mean</i>	396.022
<i>ARIMA(2,0,1)</i>	<i>with non – zero mean</i>	386.9209
<i>ARIMA(2,0,3)</i>	<i>with non – zero mean</i>	377.9495
<i>ARIMA(1,0,2)</i>	<i>with zero mean</i>	372.4052
<i>ARIMA(0,0,2)</i>	<i>with zero mean</i>	393.029
<i>ARIMA(1,0,1)</i>	<i>with zero mean</i>	383.1155
<i>ARIMA(2,0,2)</i>	<i>with zero mean</i>	374.5912
<i>ARIMA(1,0,3)</i>	<i>with zero mean</i>	372.9165
<i>ARIMA(0,0,1)</i>	<i>with zero mean</i>	396.9129
<i>ARIMA(0,0,3)</i>	<i>with zero mean</i>	394.151
<i>ARIMA(2,0,1)</i>	<i>with zero mean</i>	385.072
<i>ARIMA(2,0,3)</i>	<i>with zero mean</i>	376.1708
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<i>ARIMA (1,0,2) with zero mean</i>	<i>: 392.2688</i>	

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